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Optimal Aircraft Terrain-Following Analysis and Trajectory Generation

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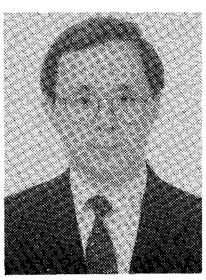
The analysis of an aircraft terrain-following (TF) problem and a numerical method for the optimal trajectory generation are the subjects of this paper. The TF problem is formulated as an optimal control problem that combines short flight time and path-following objectives. The aircraft dynamics are represented by point-mass equations of motion in a vertical plane. The problem is first studied analytically to reveal the bang-bang nature of the optimal thrust control in most cases. Then, an inverse dynamics approach is employed to solve the problem numerically. Numerical solutions are provided to support the analysis and demonstrate the features of the problem.

I. Introduction

THE key objective of aircraft terrain-following (TF) flight is to maintain a minimum set clearance above the terrain and fly fast so as to minimize the risks of being detected and tracked. Low altitude and high speed are the typical tactics. The major tasks in a TF problem are trajectory planning, guidance/control, and sensor blending for terrain data update. The issue of generating reference paths is addressed by Funk[1] where the altitude is parametrized by cubic splines. The cubic splines are determined by solving a nonlinear programming problem that minimizes the deviations from the terrain at the nodes subject to some operational constraints. An optimal control approach is taken by Menon et al.[2,3] to obtain trajectories that minimize a linear combination of flight time and terrain masking. In Ref. 4, the desired lateral motion of the aircraft is determined as the result of an optimal control problem, whereas the longitudinal path is constructed by segments of parabolas that clear critical points. By and large, only kinematics of the aircraft are used in the trajectory planning in the previous work. While ignoring the dynamics certainly helps simplify the trajectory planning, aircraft performance limitations in terms of aerodynamic and propulsion controls are not easy to enforce directly. Therefore, the optimal trajectory obtained can be overly optimistic. More significantly, the optimal solution can be quite different, depending on whether or not aircraft dynamics are included in the problem formulation. For example, in vertical planar motion, the kinematic equations are $\dot{x} = V\cos\gamma$ and $\dot{y} = V\sin\gamma$, where x and y are the downrange distance and altitude, V the velocity, and γ the flight-path angle. If V and γ are used as the controls and the performance index only depends on x and y, the optimal solution calls for constant, maximum velocity.[2,3] In contrast, as will be seen in this study, the optimal velocity is time varying when the aircraft dynamics are included and the aerodynamic and propulsion forces are used as the controls.

Once a feasible reference TF trajectory is obtained, it is the task of the TF control system to guide the aircraft to track the reference trajectory. Predictive control techniques are employed in Refs. 5 and 6 for this purpose where the aircraft is represented by linear, discrete-time models. When the flight conditions during TF vary significantly, some kind of gain scheduling or a nonlinear approach should be adopted. A nonlinear control approach called optimal decision strategy[7] is applied to the TF control problem in Ref. 8. Reference 9 takes up the same approach but discretizes the problem into a linear programming problem. For a glance of the effects of radar systems on TF flight, the reader is referred to Refs. 10 and 11.

This paper addresses the analysis and off-line planning of optimal TF trajectories. The objectives of this work are 1) to investigate the optimal TF trajectories with full point-mass nonlinear dynamics so that a better understanding of the dynamic aspect of the problem and the requirements on aerodynamic and propulsion controls for



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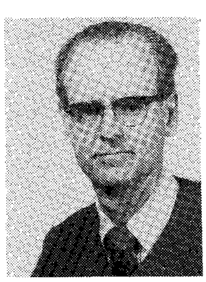

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optimal TF flight can be obtained; 2) to demonstrate the application of a new inverse dynamics approach for trajectory optimization,[12] which appears to be particularly well-suited for the TF problem; and 3) to generate reference trajectories for our second-phase study[13] in which we examine the application of a new nonlinear predictive control method to on-board guidance for TF flight. In Sec. II the TF problem is formulated as an optimal control problem that takes account of both the closeness of the trajectory to the terrain and the time of flight. An analytical investigation of the optimality of singular thrust control is presented in Sec. III. Then an inverse dynamics approach is used in Sec. IV to obtain the numerical solution to the optimal TF problem that would otherwise be rather difficult to deal with. Section V concludes the work by summarizing the achievements.

II. Formulation of Optimal Terrain-Following Problem

The TF problem usually considers motion in the vertical plane, whereas the terrain-avoidance problem deals with lateral motion.[1] Also, since the adoption of aircraft dynamics considerably increases the problem complexity, we choose to restrict our attention to two-dimensional motion at this time. However, the analysis and methodology used here can be readily extended to three-dimensional flight.

Consider the longitudinal dynamics of an aircraft over a flat Earth. The point-mass equations of motion for planar flight are

$$\dot{y} = v \sin\gamma \tag{1}$$

$$\dot{x} = v \cos\gamma \tag{2}$$

$$\dot{v} = \frac{T\cos\alpha - D}{m} - g\sin\gamma \tag{3}$$

$$\dot{\gamma} = \frac{T\sin\alpha + L}{mv} - \frac{g}{v}\cos\gamma \tag{4}$$

where the state variables are the position coordinates x and y, the velocity v, and the flight-path angle γ. The terms L and D are the aerodynamic lift and drag, respectively:

$$L = qSC_L, \qquad D = qSC_D \tag{5}$$

where S is the reference area of the aircraft and $q = \frac{1}{2}\rho v^2$ is the dynamic pressure, with the atmospheric density denoted by ρ, which is assumed to be an exponential function of the altitude y with the scale height equal to 23,800 ft. The lift and drag coefficients C_L and C_D are functions of the angle of attack α and Mach number M. The thrust T is defined by

$$T = T_{\max}(M, y)\eta \tag{6}$$

where $0 \le \eta \le 1$. The maximum thrust $T_{\max}$ is a function of Mach number M and altitude y in general. The value of η determines the required thrust level, and thus we call η the throttle setting in this specific sense. The aerodynamic and propulsion controls are then represented by α and η, respectively. We use the following dimensionless variables for better numerical properties:

$$h = \frac{gy}{v_s^2}, \qquad X = \frac{gx}{v_s^2}, \qquad V = \frac{v}{v_s} \tag{7}$$

$$\tau = \frac{gt}{v_s}, \qquad A_T = \frac{T_{\max}\eta}{mg}, \qquad A_D = \frac{D}{mg}, \qquad A_L = \frac{L}{mg} \tag{8}$$

where v_s is the speed of sound at sea level. Neglecting the change of the mass m, we have the nondimensional equations

$$h' = V\sin\gamma \tag{9}$$

$$X' = V\cos\gamma \tag{10}$$

$$V' = A_T\cos\alpha - A_D - \sin\gamma \tag{11}$$

$$\gamma' = \frac{1}{V}(A_T\sin\alpha + A_L) - \frac{\cos\gamma}{V} \tag{12}$$

where the prime indicates differentiation with respect to τ. Let the state vector $\boldsymbol{x} = (h\ X\ V\ \gamma)^T$. The trajectory of the aircraft is to satisfy boundary conditions

$$\boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{13}$$

$$\boldsymbol{\Psi}[\boldsymbol{x}(\tau_f)] = 0 \tag{14}$$

where initial conditions are specified. The final time τ_f is free, and $\boldsymbol{\Psi}[\boldsymbol{x}(\tau_f)]$ represents a system of $r \le 4$ algebraic equations that specify the terminal manifold. The constraints on the controls are

$$0 \le \eta \le 1.0, \qquad -90\ \text{deg} < \alpha_{\min} \le \alpha \le \alpha_{\max} < 90\ \text{deg} \tag{15}$$

Suppose that the terrain is given as a function of the downrange distance X. Let $F(X)$ be the terrain elevation plus the set clearance at X. No matter what the actual terrain is, the function $F(X)$ can always be made sufficiently differentiable and avoid any points where $dF(X)/dX = \pm\infty$. With a slight abuse of terminology, $F(X)$ will be referred to as the "terrain" in what follows. The optimal TF problem is to find the controls $\eta^*(\tau)$ and $\alpha^*(\tau)$ such that along a trajectory of the aircraft [Eqs. (9–12)] satisfying Eqs. (13–15), the performance index

$$J = \phi\tau_f + (1-\phi)\int_0^{\tau_f}[h - F(X)]^2\,d\tau \qquad 0 < \phi \le 1.0 \tag{16}$$

is minimized. The solution for $0 < \phi < 1$ is called the optimal TF trajectory. It provides a time history $\boldsymbol{x}(\tau)$ within the performance range of the aircraft. It should be pointed out that the second term in Eq. (16) can be zero only in the very special case where the aircraft trajectory at $t = 0$ tangentially touches the terrain. Otherwise, the integral will not be zero. Note also that a nonzero ϕ in the performance index (16) is essential, because although the aircraft may follow a given terrain under many control pairs $(\eta(\tau),\ \alpha(\tau))$, the optimal trajectory should use the shortest possible time-of-flight to reduce the risks of being detected and exposed to threats. In Ref. 8 a constant-energy trajectory in the time domain is constructed based on the given terrain. This does not account for the time-of-flight factor, and the constant-energy requirement may in fact ask for very low aircraft speed if the terrain elevation increases dramatically. As a result, flying the reference trajectory may exceed the performance capabilities of the aircraft.

When $\phi = 1$, the problem becomes a minimum-time problem. In this case a state inequality constraint

$$h - F(X) \ge 0 \tag{17}$$

should be imposed to prevent the aircraft from intersecting the terrain.

III. Analysis of Optimal Thrust Control

By the minimum principle,[14] the fact that the throttle control η appears linearly in the state equations (9–12) dictates that the optimal η^* either takes upper and/or lower bounds (bang-bang type) or, for a finite interval in $[0, \tau_f]$, it continuously varies within the bounds (singular control). Although the problem ultimately has to be solved numerically, an analytical study of the possibility of singular arcs not only offers insight into the problem that cannot be obtained by numerical solutions but also provides valuable guidance to the selection of numerical procedures so that a more accurate optimal solution can be obtained. The question of whether or not singular throttle control for the optimal TF problem exists is answered in a more general context as stated in the following theorem.

Theorem (*nonexistence of singular arcs*). Consider a class of aircraft models for which the lift and drag coefficients take the forms

$$C_L = \sum_{k=1}^{K} C_{L_{2k-1}}\alpha^{2k-1}$$

$$K \ge 1, \qquad C_{L_1} > 0, \qquad C_{L_{2k-1}} \ge 0, \qquad k = 2, \ldots, K \tag{18}$$

$$C_D = \sum_{n=0}^{N} C_{D_{2n}}\alpha^{2n}$$

$$N \ge 0, \qquad C_{D_{2n}} \ge 0, \qquad n = 0, 1, 2, \ldots, N \tag{19}$$

where K and N are positive integers and all the coefficients $C_{L_{2k-1}}$ and $C_{D_{2n}}$ are functions of Mach number. The propulsion force is modeled by Eq. (6). The state equations are Eqs. (9–12). The performance index of minimization is

$$J = \phi\tau_f + (1-\phi)\int_0^{\tau_f}[L(h, X)]^2\,\mathrm{d}\tau, \qquad 0 < \phi \le 1.0 \tag{20}$$

where $L(h, X)$ is any given function of class C^1, dependent only on h and X. Assume that an optimal solution subject to control constraints Eq. (15) and boundary conditions Eqs. (13) and (14) exists. Then, the optimal throttle $\eta^*(\tau)$ must be of the bang-bang type in any finite interval in $[0, \tau_f]$ where the optimal $\alpha^*(\tau)$ is interior, i.e., $\alpha_{\min} < \alpha^*(\tau) < \alpha_{\max}$.

Proof. From the Hamiltonian of the system (9–12) and (20):

$$\begin{aligned} H &= p_h V\sin\gamma + p_X V\cos\gamma + p_V(A_T\cos\alpha - A_D - \sin\gamma) \\ &\quad + p_\gamma\left(\frac{1}{V}(A_T\sin\alpha + A_L) - \frac{\cos\gamma}{V}\right) + (1-\phi)L^2(h, X) \\ &\triangleq H_1 + (1-\phi)L^2(h, X) \end{aligned} \tag{21}$$

where $\boldsymbol{p} = (p_h\; p_X\; p_V\; p_\gamma)^T$ is the costate vector that satisfies the adjoint equations of the system (9–12), $\boldsymbol{p}' = -\partial H/\partial \boldsymbol{x}$. In particular,

$$\begin{aligned} p_V' &= -\frac{\partial H}{\partial V} = -p_h\sin\gamma - p_X\cos\gamma - p_V\left(\cos\alpha\frac{\partial A_T}{\partial V}\right. \\ &\quad \left. - \frac{\partial A_D}{\partial V}\right) + \frac{p_\gamma}{V}\left[\frac{1}{V}(A_T\sin\alpha + A_L) - \sin\alpha\frac{\partial A_T}{\partial V}\right. \\ &\quad \left. - \frac{\partial A_L}{\partial V} - \frac{\cos\gamma}{V}\right] \end{aligned} \tag{22}$$

$$\begin{aligned} p_\gamma' &= -\frac{\partial H}{\partial \gamma} = -p_h V\cos\gamma + p_X V\sin\gamma \\ &\quad + p_V\cos\gamma - p_\gamma\frac{\sin\gamma}{V} \end{aligned} \tag{23}$$

Since the right-hand sides of Eqs. (9–12), the integrand $L(h, X)$ in the performance index, and the terminal constraint residual $\boldsymbol{\Psi}$ in (14) all do not depend explicitly on time, the Hamiltonian is constant along the optimal trajectory,[14]

$$H = H_1 + (1-\phi)L^2(h, X) = -\phi \tag{24}$$

Suppose that $[\tau_1, \tau_2] \subset [0, \tau_f]$ is a finite interval where α^* is interior. The optimality condition for α in $[\tau_1, \tau_2]$ requires

$$\begin{aligned} \frac{\partial H}{\partial \alpha} &= -p_V\left(A_T\sin\alpha + \frac{\partial A_D}{\partial \alpha}\right) \\ &\quad + p_\gamma\frac{(A_T\cos\alpha + \partial A_L/\partial\alpha)}{V} = 0 \end{aligned} \tag{25}$$

Now, suppose that a singular arc for η exists in this interval. It is thus necessary that the switching function for η be identically zero in this interval, that is,

$$p_V\cos\alpha + p_\gamma\sin\alpha/V \equiv 0 \tag{26}$$

One possible solution to Eqs. (25) and (26) is that p_V and p_γ are simultaneously zero in this interval. But $p_V = p_\gamma \equiv 0$ leads from Eqs. (22) and (23) to the system

$$-p_h\sin\gamma - p_X\cos\gamma = 0 \tag{27}$$

$$-p_h V\cos\gamma + p_X V\sin\gamma = 0 \tag{28}$$

which has a unique solution $p_h = p_X = 0$ for $V \neq 0$. Therefore, the complete costate vector is identically zero in this interval. Thus, Eq. (24) now reduces to

$$(1-\phi)L^2(h, X) = -\phi \tag{29}$$

Since $0 < \phi \le 1$, no values of L can satisfy Eq. (29). We conclude that p_V and p_γ cannot be simultaneously zero for a finite period. This implies that the determinant of the coefficient matrix of the linear algebraic system (25) and (26) in p_V and p_γ must be zero, which results in

$$A_T + \sin\alpha\frac{\partial A_D}{\partial\alpha} + \cos\alpha\frac{\partial A_L}{\partial\alpha} = 0 \tag{30}$$

Since the thrust acceleration $A_T \ge 0$, we need

$$\sin\alpha\frac{\partial A_D}{\partial\alpha} + \cos\alpha\frac{\partial A_L}{\partial\alpha} \le 0 \tag{31}$$

for Eq. (30) to hold. Evaluating the partial derivatives in Eq. (31) with C_L and C_D given in Eqs. (18) and (19) and noting that $\cos\alpha > 0$ for $|\alpha| < 90$ deg, we have

$$\begin{aligned} &0.5\left[C_{L_1} + 3C_{L_3}\alpha^2 + \cdots + (2K-1)C_{L_{2K-1}}\alpha^{2K-2}\right] \\ &\quad + \alpha\,\tan\alpha\left(C_{D_2} + 2C_{D_4}\alpha^2 + \cdots + NC_{D_{2N}}\alpha^{2N-2}\right) \le 0 \end{aligned} \tag{32}$$

Because α tan α is an even function for $|\alpha| < 90$ deg and all the coefficients in Eq. (32) are nonnegative (also $C_{L_1} > 0$), inequality (32) cannot be satisfied by any admissible α. This contradiction rules out the possibility of Eq. (26) holding in $[\tau_1, \tau_2]$. It follows that no singular arcs exists for the optimal throttle control in this interval.

Discussion:

1) For the case where there is only a single control, there exists the so-called generalized Legendre–Clebsch condition that is necessary for the optimal singular control to be present (e.g., Ref. 15). But for multicontrol systems like the current one, a case-by-case study will have to be conducted to determine the optimality of singular control.

2) The conclusion of nonsingular optimal throttle control applies to a class of optimal control problems with the performance index given by Eq. (20), including the TF problem when $L(h, X) = h - F(X)$. When the time of flight is not a concern [$\phi = 0$ in the performance index and $L(h, X) = h - F(X)$], we call the problem a pure terrain-following (PTF) problem. For a PTF problem, Eq. (29) can hold true in an interval where $h \equiv F(X)$, and Eq. (26) will not cause contradiction. (In such a case, all the necessary conditions for optimality are still satisfied, in spite of the fact that the costate vector $\boldsymbol{p} = 0$ in that interval.) In other words, singular throttle control is possible for the PTF problem only when the terrain is perfectly followed. A numerical solution in the next section confirms this observation.

3) It is interesting to note that this bang-bang throttle control property is not dependent on the specifics of the terrain function $F(X)$ or the boundary conditions. The number of throttle control switchings, on the other hand, will probably be related to the particular terrain assumed and the boundary conditions, although a rigorous proof is not available yet. The bang-bang thrust program also indicates that the velocity will not be a constant in general. This contrasts with the constant-velocity solution obtained when only kinematics are used.[2,3]

IV. Numerical Solutions by an Inverse Dynamics Approach

Although the analysis in the preceding section has revealed some insight to the TF problem, a complete solution still has to be found numerically by either direct or indirect approaches. Given that the altitude is the most important variable to control in a TF problem, it appears that the inverse dynamics approach proposed in Ref. 12 is well suited for the trajectory optimization purpose. Since the terrain $F(X)$ is given as a function of the (dimensionless) downrange distance X, we find it more convenient to use X as the independent variable. The new state equations from Eqs. (9–12) are now

$$\frac{\mathrm{d}h}{\mathrm{d}X} = \tan\gamma \tag{33}$$

$$\frac{\mathrm{d}V}{\mathrm{d}X} = (A_T\cos\alpha - A_D - \sin\gamma)\frac{1}{V\cos\gamma} \tag{34}$$

$$\frac{\mathrm{d}\gamma}{\mathrm{d}X} = (A_T\sin\alpha + A_L - \cos\gamma)\frac{1}{V^2\cos\gamma} \tag{35}$$

The inverse dynamics approach proceeds as follows: Given a specified twice continuously differentiable altitude profile $z(X)$, let $h(X) = z(X)$. The required flight-path angle from Eq. (33) is then

$$\gamma = \tan^{-1}\left(\frac{dz}{dX}\right) \tag{36}$$

With the aid of Eq. (35), we differentiate Eq. (36) with respect to X once to arrive at

$$A_T \sin\alpha + A_L - \left(1 + V^2\cos^2\gamma\frac{d^2z}{dX^2}\right)\cos\gamma = 0 \tag{37}$$

Suppose that a history for η is also specified. At each step of the integration of the system equations, Eq. (37) constitutes a nonlinear algebraic equation in α that is required for $h(X) = z(X)$. A Newton iteration scheme is very effective to solve for α, provided an $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ exists to satisfy Eq. (37). The obtained α is then used in the integrator. If the optimal altitude profile $z^*(X)$ together with an optimal $\eta^*(X)$ is found to optimize the performance index, the optimal $\alpha^*(X)$ is obtained from Eq. (37) and the optimal control problem is solved.

It should be noted that the above problem constitutes essentially a system of differential algebraic equations (DAEs) of index 1. Extensive work has been undertaken on numerical solutions of DAEs (e.g., Refs. 16–18). The standard technique combines integration and the simultaneous solution of the algebraic equations. It would be a worthwhile effort to develop a general trajectory optimization program that employs the concept of inverse dynamics and state-of-the-art algorithms for DAEs. However, for our particular purpose here, the inverse dynamics sequential approach described above works well and requires relatively few modifications in our existing trajectory optimization software.

We choose to represent $h(X) = z(X)$ by cubic splines. In addition to having the required smoothness, cubic splines minimize the integral

$$\int_0^{x_f}\left[\frac{d^2h}{dX^2}\right]^2 dX \tag{38}$$

among all C^2 functions $f(X)$ that pass through a given set of pairs $\{(0, h_0), \ldots, (X_f, h_f)\}$ and satisfy $df(0)/dX = dF(0)/dX$ and $df(X_f)/dX = dF(X_f)/dX$.[19] Since d^2h/dX^2 is proportional to the normal acceleration along the path, the minimization of the integral (38) is desirable for reducing both aircraft crew discomfort and airframe stress. The throttle history $\eta(X)$ is first parametrized by piecewise-linear continuous functions. The purpose here is to find the switching structure of the optimal bang-bang η control. Once the number of switchings is determined, η is parametrized by a piecewise-constant function that has the same switching structure as the optimal η^*. The node values of the altitude cubic splines and the throttle piecewise-linear functions (or the switching points in the piecewise-constant parametrization case) are the optimization parameters to be chosen to minimize the performance index

$$j = \int_0^{X_f}\left\{\phi\left(\frac{1}{V\cos\gamma}\right) + (1-\phi)[h - F(X)]^2\right\} dX \qquad 0 < \phi < 1.0 \tag{39}$$

where the integral of the first term in the performance index is the time of flight multiplied by ϕ. Through the integration of the system equations (33–35), the performance index becomes a function of the optimization parameters. Thus, we must solve a nonlinear programming problem. A sequential quadratic programming (SQP) algorithm[20] is used to solve it.

Compared to the traditional nonlinear programming approach in which the controls are parametrized, the inverse dynamics approach used here has a more direct influence on the shape of the altitude profile, which is the most critical quantity to control in a TF problem. This direct influence is the key to making the optimization problem better conditioned and enhancing the convergence of the optimization process. Similarly, for a minimum-time problem where a state inequality constraint (17) is imposed, the inverse dynamics approach makes it particularly simple and effective to enforce the constraint. Although an algebraic equation (37) has to be solved iteratively at each step, the iteration typically converges in one or two cycles. And as a result of the inverse dynamics procedure, the h and γ equations (33) and (35) do not need to be integrated. Instead, $h(X) = z(X)$ and γ is obtained from Eq. (36). So the overall efficiency of the numerical process is not compromised. An added advantage of the direct parametrization of $h(X)$ is that any terminal conditions of the form of $h(X_f) = h_f$ and $\gamma(X_f) = \gamma_f$ can be handled easily by specifying the value $h(X_f)$ and the slope $dh(X_f)/dX$ in the parametrization of the cubic splines that define $h(X)$. Then these two constraints are automatically satisfied. This feature tends to speed up the convergence of the optimization process significantly. It should be mentioned that collocation methods[21] that parametrize both states and controls and satisfy the system equations at specified time nodes have similar advantages of robustness and good conditioning. Since the inverse dynamics method numerically integrates the equations of motion, the number of optimization parameters can be significantly reduced, and the obtained trajectory always satisfies the system equations accurately.

For our numerical experiments, a terrain function $F(X)$ given in Ref. 22 is digitized and then interpolated by cubic splines. The aircraft model is for a supersonic fighter aircraft, taken from Ref. 23 (airplane 2). The lift and drag coefficients are given as

$$C_L = C_{L\alpha}\alpha \tag{40}$$

$$C_D = C_{D0} + C_{L\alpha}\alpha^2 \tag{41}$$

where $C_{L\alpha}$ and C_{D0} are originally given in tabulated data as functions of Mach number M and are fitted by least-squares polynomials for $0 \le M \le 2.4$ in this study:

$$C_{L\alpha} = 2.2371 + 0.34008M - 0.2615M^2 + 0.01085M^3 \tag{42}$$

$$C_{D0} = 0.0065 - 0.0022187M - 0.01649M^2 + 0.043848M^3 - 0.028402M^4 + 0.005493M^5 \tag{43}$$

Note that the C_L and C_D models (40) and (41) fall into the class described by Eqs. (18) and (19), and $C_{L\alpha}$ and C_{D0} fitted by Eqs. (42) and (43) are positive for $0 \le M \le 2.4$. In this study the bounds on α are set at

$$\alpha_{\max} = 20\,\text{deg}, \qquad \alpha_{\min} = -10\,\text{deg} \tag{44}$$

The maximum thrust $T_{\max}$ of the aircraft is given in tabular form as a function of altitude and Mach number in Ref. 23. Linear interpolations are used here for table look-up evaluations of $T_{\max}$. We realize that the optimization problem is not globally twice continuously differentiable because of the linear interpolations. Two-dimensional smooth interpolations could be used to make the optimization problem globally C^2. This would significantly increase the computation time, however, and was found unnecessary. Because the data table has a sparse grid, the linearly interpolated problem remains C^2 in a reasonably large neighborhood of the optimum point. The SQP algorithm experienced on difficulty in finding the solution.

The weight of the aircraft is set at 34,000 lb and is considered constant. In terms of the dimensional variables, the terminal conditions of the trajectory are tightly constrained to be

$$y(X_f) = 640\,\text{ft}, \qquad \gamma(X_f) = 0\,\text{deg}$$

$$v(X_f) = 1114.9\,\text{ft/s} \qquad (M = 1.0) \tag{45}$$

where $y(X_f) = F(X_f)$ and $\gamma(X_f) = 0$ deg gives $dy(X_f)/dX = dF(X_f)/dX = 0$. As mentioned before, the first two constraints in Eq. (45) are automatically satisfied by specifying the value and slope of the cubic splines defining $h(X)$ at the last node. This leaves only the constraint on the final v to be met.

The optimal solutions corresponding to various combinations of initial conditions have been obtained. For instance, for $y(0) = 200$ ft and $\gamma(0) = 0$ deg, the aircraft trajectory starts on the terrain function

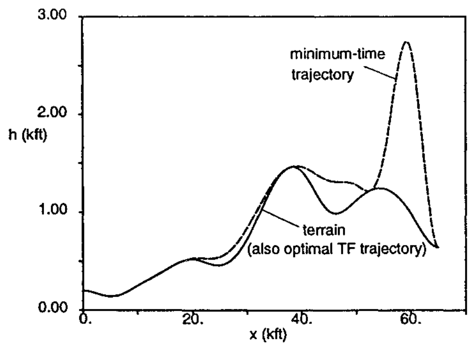

Fig. 1 Optimal TF and minimum-time altitude histories (optimal TF trajectory coincides with terrain).

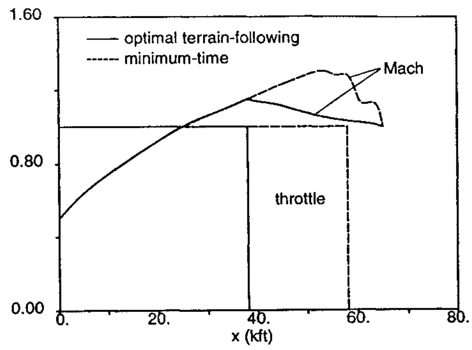

Fig. 2 Mach number and throttle histories.

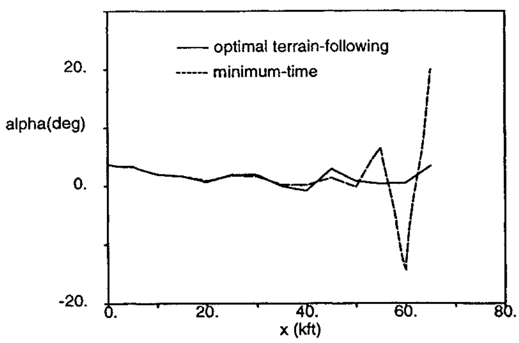

Fig. 3 Angle-of-attack histories.

$F(X)$ and is tangential to $F(X)$. Figure 1 shows for $v_0 = 557.5$ ft/s ($M = 0.5$) the altitude histories of the optimal TF trajectory (with $\phi = 0.5$) and the minimum-time trajectory [$\phi = 1$; subject to constraint (17)], where the state inequality constraint (17) is converted to a terminal constraint by a standard transformation

$$\zeta(X_f) \geq 0 \tag{46}$$

$$\frac{d\zeta}{dX} = \min\{0,\ h - F(X)\}, \qquad \zeta(0) = 0 \tag{47}$$

The optimal TF trajectory is indiscernible from $F(X)$ because the terrain is followed exactly. The minimum-time trajectory, on the other hand, has large overshoot above the terrain after the aircraft clears the highest point of the terrain; this is found to be typical of the minimum-time solutions. The final dive maneuver on the minimum-time trajectory is a coasting arc with $\eta = 0$. The main function of this dive is to reduce the velocity to meet the terminal velocity constraint. This phenomenon can be clearly seen from Fig. 2, which shows the variations of Mach number and throttle setting along the optimal TF and minimum-time trajectories. Figure 3 contains the corresponding α histories. The corners in the α plots are due to the third-order derivative discontinuities in the cubic spline parametrization of $h(X)$. The time of flight on the minimum-time trajectory is 61.1 s, whereas the time of flight is 63.4 s on the optimal TF trajectory. It should be pointed out that for the given initial conditions, the value of ϕ does not affect the optimal TF solution as long as $\phi \in (0,\ 1)$,

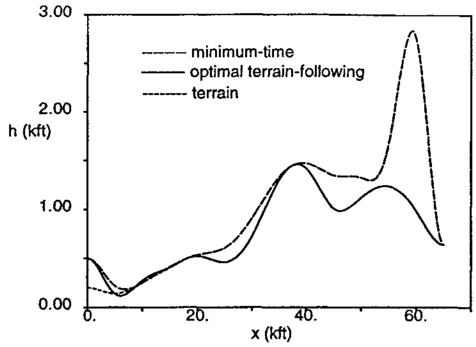

Fig. 4 Optimal TF and minimum-time altitude histories.

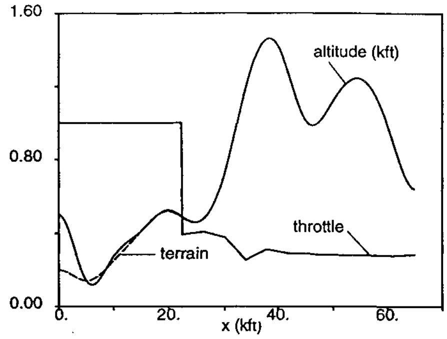

Fig. 5 Altitude and throttle histories along PTF trajectory.

because the second term in the performance index (39) is always zero along the optimal trajectory.

It turns out that for the given initial conditions, terrain, and aircraft model, many perfect TF trajectories exist, all with different pairs of control histories $[\eta(X),\ \alpha(X)]$ and time of flight. That is, many solutions exist for the PFF problem ($\phi = 0$). [For $\phi = 0$, the performance index (38) attains its minimum value of zero along any of the perfect TF trajectories for the given $y(0)$ and $\gamma(0)$. We can easily find variable throttle histories that can achieve $J = 0$ in this case. By the proof of the theorem in Sec. III, this can only be a degenerate case in optimal control where the costate $\boldsymbol{p} \equiv 0$ in the entire interval of optimal solution.] We have found, among the many solutions, a perfect TF trajectory that requires 101 s of flight. This highlights the importance of the performance index (16) [or (39)] with $\phi \in (0,\ 1)$: It strikes a balance between the two objectives of short time of flight and TF.

For another set of initial conditions,

$$y(0) = 500\,\text{ft}, \qquad \gamma(0) = 0\,\text{deg}$$
$$v(0) = 557.5\,\text{ft/s} \qquad (\text{M} = 0.5) \tag{48}$$

the trajectory starts 300 ft above the terrain. Figure 4 shows the altitude histories for the two types of trajectories for $\phi = 0.5$ and $\phi = 1$ [subject to constraint (17)]. The optimal TF trajectory tracks the terrain as expected after an initial transient period. The minimum-time trajectory exhibits the same characteristics of large deviations from the terrain toward the end of the trajectory, as observed in Fig. 1. Interestingly, the minimum-time trajectory also starts with a dive. The times of flight are 63 and 60.7 s, respectively.

In all the tests that we have done for the cases where $\phi \in (0,\ 1]$, the optimal α^* is interior, and only a single switching in the optimal bang-bang throttle histories has been found (all from full to zero throttle). This switching structure, of course, may not be the only possibility, as noted at the end of Sec. III.

Also concluded in the analysis of Sec. III is that for a PTF problem in which $\phi = 0$, variable (singular) throttle control is possible when the terrain is tracked perfectly. For the same initial conditions (48), the PTF problem is solved. In Fig. 5 the altitude history and the throttle control history are plotted together. It is seen that indeed the throttle control is on the upper bound when the trajectory is not on the terrain. Once the trajectory begins to follow the terrain exactly, an unmistakable singular arc appears in the throttle history. The time

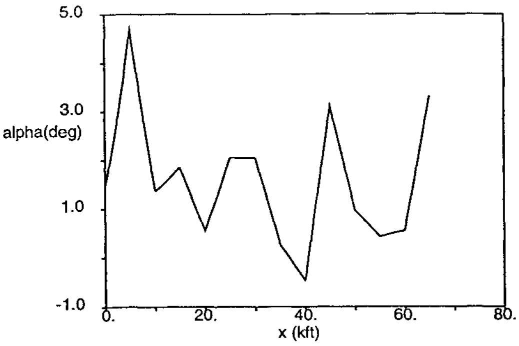

Fig. 6 Angle-of-attack history along PTF trajectory.

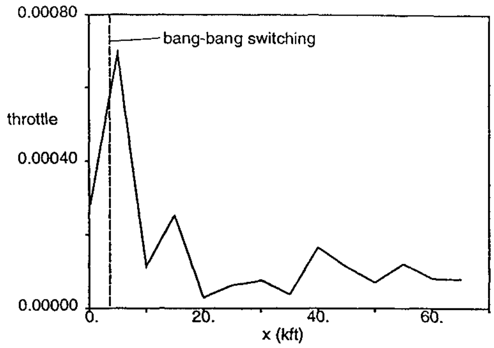

Fig. 7 Comparison of nonoptimal and optimal (bang-bang) throttle histories.

of flight is 66.2 s, compared to 63 s and 60.7 s of the optimal TF and minimum-time solutions. The α history along this PTF trajectory is shown in Fig. 6.

In the above problems, the number of optimization parameters ranges from 15 to 26, depending on the problem. With the inverse dynamics approach, the optimization process converges typically within 30 iterations, which allowed us to investigate many cases very reliably and efficiently. Only a small fraction of the results are reported here.

As a final note on how the analytical study may be used to verify the validity of the numerical solutions and improve the numerical procedure, we present an optimal TF problem in which the initial conditions are

$$y(0) = 200\,\text{ft}, \qquad \gamma(0) = 0\,\text{deg}$$

$$v(0) = 1616.6\,\text{ft/s} \qquad (M = 1.45) \tag{49}$$

The final conditions are still the ones given in Eq. (45). When the problem with $\phi = 0.5$ is solved with η being parametrized by piecewise-linear continuous functions, the terrain is followed perfectly with a time of flight of 49.8 s. The converged throttle history looks distinctively singular, as seen in Fig. 7. By the theorem in Sec. III, however, we know that this cannot be the true optimal solution. This happened because the optimization algorithm finds the best throttle history within the class of piecewise-linear continuous functions with the given discretization grid. In this case, the result is significantly different from the true optimal throttle control, which belongs to the class of functions that are only piecewise continuous. By allowing bang-bang control as stated by the theorem, we do obtain a bang-bang throttle history that also enables the aircraft to follow the terrain perfectly with a time of flight of 48.7 s, reducing the time of flight by more than a full second. The point where the optimal throttle history switches from full to zero throttle is also indicated in Fig. 7.

V. Conclusion

The aircraft TF problem is formulated as an optimal control problem with full point-mass nonlinear aircraft dynamics. The performance index is such that two critical measures in TF flight, closeness to the terrain and quickness of the flight, are optimized in a balanced way. For a rather general class of aircraft models, an analysis of the singular control for the TF problem reveals that the optimal thrust control can only be of the bang-bang type when the optimal angle of attack is interior; if the time of flight is not included in the performance index, the optimal thrust may be variable (singular) only when the terrain is followed exactly. A inverse dynamics approach is used to solve the optimal TF problem numerically. This approach simplifies this otherwise quite difficult problem considerably. With the numerical procedure guided by the analysis, the generation of accurate optimal TF trajectories can be carried out reliably. Numerical solutions for a supersonic fighter aircraft demonstrate the features of the optimal TF trajectories and support the analysis.

References

[1]Funk, J. E., "Optimal-Path Precision Terrain-Following System," *Journal of Aircraft*, Vol. 14, No. 2, 1977, pp. 128–134.

[2]Menon, P. K. A., Kim, E., and Cheng, V. H. L., "Optimal Trajectory Synthesis for Terrain-Following Flight," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 4, 1991, pp. 807–813.

[3]Menon, P. K. A., and Cheng, V. H. L., "Minimum-Exposure Near-Terrain Flight Trajectories for Rotorcraft," *Control and Dynamic Systems*, edited by C. T. Leondes, Vol. 52, Academic, San Diego, CA, 1992, pp. 391–433.

[4]Asseo, S. J., "Terrain Following/Terrain Avoidance Path Optimization Using the Method of Steepest Descent," *Proceedings of the IEEE National Aerospace and Electronics Conference*, Vol. 3, Inst. of Electrical and Electronics Engineers, New York, 1988, pp. 1128–1136.

[5]Hess, R. A., and Jung, Y. C., "An Application of Generalized Predictive Control to Rotorcraft Terrain-Following Flight," *IEEE Transactions on Systems, Man, and Cybernetics*, Vol. 19, No. 5, 1989, pp. 955–962.

[6]Reid, J. G., Chaffin, D. E., and Silverthorn, J. T., "Output Predictive Algorithmic Control: Precision Tracking with Application to Terrain Following," *Journal of Guidance and Control*, Vol. 4, No. 5, 1981, pp. 502–509.

[7]Spong, M. W., Thorp, J. S., and Kleinwaks, J. M. M., "The Control of Robot Manipulators with Bounded Input," *IEEE Transactions on Automatic Control*, Vol. AC-31, 1986, pp. 483–490.

[8]Lee, S. M., Bien, Z., and Park, S. O., "On-Line Optimal Terrain-Tracking System," *Optimal Control Applications and Methods*, Vol. 11, No. 4, 1989, pp. 289–306.

[9]Rehbock, V., Teo, K. L., and Jennings, L. S., "A Linear Programming Approach to On-Line Constrained Optimal Terrain-Tracking Systems," *Optimal Control Applications and Methods*, Vol. 14, No. 4, 1993, pp. 229–241.

[10]Bair, G. L., and Johnston, D. A., "Radar Systems Analysis Using DTED Data," *Proceedings of the IEEE National Aerospace and Electronics Conference*, Inst. of Electrical and Electronics Engineers, New York, 1987.

[11]Zelenka, R. E., "Design and Analysis of a Kalman Filter for Terrain-Reference Positioning and Guidance," *Journal of Aircraft*, Vol. 31, No. 2, 1994, pp. 339–344.

[12]Lu, P., "Inverse Dynamics Approach to Trajectory Optimization for an Aerospace Plane," *Journal of Guidance, Control, and Dynamics*, Vol. 16, No. 4, 1993, pp. 726–732.

[13]Lu, P., and Pierson, B. L., "Aircraft Terrain-Following Based on a Nonlinear Continuous Predictive Control Approach," *Journal of Guidance, Control, and Dynamics* (to be published); also AIAA Paper 95-3342, Aug. 1995.

[14]Bryson, A. E., and Ho, Y. C., *Applied Optimal Control*, Hemisphere, New York, 1975.

[15]Kelley, H. J., Kopp, R. E., and Moyer, A. G., "Singular Extremals," *Topics in Optimization*, edited by G. Leitmann, Vol. 2, Academic, New York, 1966, Chap. 3.

[16]Gear, C. W., "Simultaneous Solution of Differential-Algebraic Equations," *IEEE Transactions on Circuit Theory*, Vol. 18, 1971, pp. 89–95.

[17]Hairer, E., Lubich, C., and Roche, M., *The Numerical Solution of Differential-Algebraic Systems by Runge–Kutta Methods*, Lecture Notes in Mathematics, Vol. 1409, Springer–Verlag, New York, 1989.

[18]Brenan, K. E., Campbell, S. L., and Petzold, L. R., *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*, Elsevier Science, New York, 1989.

[19]Stoer, J., and Bulirsch, R., *Introduction to Numerical Analysis*, Springer–Verlag, New York, 1980.

[20]Pouliot, M. R., "CONOPT2: A Rapidly Convergent Constrained Trajectory Optimization Program for TRAJEX," General Dynamics, Rept. GDC-SP-82-008, Convair Div., San Diego, CA, 1981.

[21]Hargraves, C. R., and Paris, S. W., "Direct Trajectory Optimization Using Nonlinear Programming and Collocation," *Journal of Guidance, Control, and Dynamics*, Vol. 10, No. 4, 1987, pp. 338–342.

[22]Funk, J. E., "Terrain-Following Control Based On an Optimal Spline Model of Aircraft Motion," Ph.D. Dissertation, Dept. of Aerospace Engineering, Univ. of Michigan, Ann Arbor, MI, 1976.

[23]Bryson, A. E., and Desai, M. N., "Energy-State Approximation in Performance Optimization of Supersonic Aircraft," *Journal of Aircraft*, Vol. 6, No. 6, 1969, pp. 481–488.